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# A relativistically covariant version of Bohm's quantum field theory for the scalar field

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**Abstract**

We give a relativistically covariant, wave-functional formulation of Bohm's quantum field theory for the scalar field based on a general foliation of spacetime by space-like hypersurfaces. The wave functional, which guides the evolution of the field, is spacetime-foliation independent but the field itself is not. Hence, in order to have a theory in which the field may be considered a beable, some extra rule must be given to determine the foliation. We suggest one such rule based on the eigenvectors of the energy–momentum tensor of the field itself.

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## 1. Introduction

In this paper, we begin by deriving a suitable expression for a scalar Hamiltonian of a real classical scalar field on a space-like hypersurface. This is an essential difference from Bohm's starting point which takes the time component of the total four-momentum as the Hamiltonian and implicitly assumes the space components are zero. This implicitly introduces a preferred frame of reference and hence a non-covariant theory. It is therefore not surprising that Bohm finds the vacuum field to be stationary in one particular frame only. In addition, an arbitrary foliation of flat spacetime by the equal-time hyperplanes of the chosen frame was tacitly assumed. Such a procedure obviously conflicts with the requirements of a relativistically invariant theory. Formulating the theory in a fully covariant manner makes apparent the dependence of the field evolution on the foliation of spacetime and emphasizes that this has nothing to do with the arbitrary frame of reference used to describe the system. We proceed to quantize the field on a general hypersurface and to develop the form of the wave functionals of the quantized field. We show how to apply the Hamiltonian density to the wave functional and give two explicit examples. We then demonstrate how one may integrate the equations of motion to calculate the evolution of the Bohm field over an arbitrary family of space-like hypersurfaces. Noting that the foliation dependence of the field is somewhat at odds with the desire to produce a theory of beables, it is clear that some extra rule is required to determine the foliation[1]. We propose that, given some 'initial' space-like hypersurface, and the field thereon, the foliation may be determined in a natural way using the flows of energy–momentum determined by the field itself.

## 2. Covariant formulation of quantum field theory

As shown by Schwinger [2] the evolution of a state vector $\Psi[\sigma]$ on a space-like hypersurface $\sigma$ is given by an equation of the form

$$\mathrm{i}\frac{\delta\Psi[\sigma]}{\delta\sigma(x)} = \mathcal{H}(x)\Psi[\sigma] \tag{1}$$

where $\mathcal{H}(x)$ is an invariant (scalar) function of the field quantities[2]. This result guarantees independence of the evolution with respect to foliation by a family of space-like hypersurfaces $(\sigma)$. Assuming a continuous family $\sigma$ ordered by a scalar time parameter $t$ (only a label!), one may integrate the equation over one of the surfaces to give

$$\mathrm{i}\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \int_\sigma \sqrt{-g}\,\mathrm{d}^3x\,\mathcal{H}(x)\Psi. \tag{2}$$

One may obtain a suitable energy density $\sqrt{-g}\mathcal{H}$, proceeding classically at first, by considering the action principle with variation of the boundaries of a four-dimensional region and the field quantities [3, 4]. One takes as action $(I)$

$$I = \int_R \sqrt{-g}\,\mathrm{d}^4x\,L \tag{3}$$

where $L$ is the Lagrangian and $R$ is a four-dimensional region of spacetime. Denoting the unit normal 4-vector to the space-like hypersurface $(\sigma)$ by $n_\mu$, the variation in $R$ may be written as

$$\mathrm{d}\sigma_\mu\,\mathrm{d}x^\nu = \delta\Omega n_\mu n^\nu \tag{4}$$

where $\mathrm{d}x^\nu$ is a displacement normal to $\sigma$ and $\delta\Omega$ is the magnitude of the change in four volume. Hence,

$$\delta\int_R \sqrt{-g}\,\mathrm{d}^4x\,L = \int_R \sqrt{-g}\,\mathrm{d}^4x\,\delta L + \int_B L\,\mathrm{d}\sigma_\mu\,\mathrm{d}x^\nu \tag{5}$$

where $B$ is the boundary of $R$. Assuming that the field equations are satisfied by the field quantities $(\phi^\alpha)$ one gets

$$\delta I = \int_B \mathrm{d}\sigma_\mu \left[\frac{\partial L}{\partial(\partial_\mu\phi^\alpha)}\delta\phi^\alpha + \left(g^\mu_\nu L - \partial_\nu\phi^\alpha\frac{\partial L}{\partial(\partial_\mu\phi^\alpha)}\right)\mathrm{d}x^\nu\right] \tag{6}$$

which gives

$$\frac{\delta I}{\delta\phi^\alpha} = n_\mu\frac{\partial L}{\partial(\partial_\mu\phi^\alpha)} = \pi_\alpha \tag{7}$$

where $\pi_\alpha$ is the momentum and

$$\frac{\delta I}{\delta\Omega} = L - (n^\nu\partial_\nu\phi^\alpha)\,\pi_\alpha \tag{8}$$

[1] Duerr *et al* [1] have previously discussed the need for some additional structure to determine the required foliation of spacetime in the context of their Bohm–Dirac particle trajectory model.
[2] Schwinger goes on to make a special choice for $\mathcal{H}(x)$ which yields what he then refers to as the interaction representation. We do not follow him in this but stay with general condition (1).

and

$$\sqrt{-g}\mathcal{H} = -(\sqrt{-g})\frac{\delta I}{\delta \Omega}. \tag{9}$$

No restriction on the foliation is required[3].

## 3. Real scalar field

In the case of a real scalar field the energy density is given by

$$\sqrt{-g}\mathcal{H} = \sqrt{-g}\tfrac{1}{2}[(n^\lambda \partial_\lambda \phi)^2 + \partial_j \phi \partial^j \phi + m^2 \phi^2] \tag{10}$$

with $n^\lambda \partial_\lambda \phi = \pi$ the momentum[4]. The metric tensor $g_{\mu\nu}$ adapted to the slicing of space and time in terms of a lapse function $N$ and shift vectors $N^j$ is given by

$$[g_{\mu\nu}] = \begin{bmatrix} N^2 - N_S N^S & -N_k \\ -N_i & -{}^{(3)}g_{ik} \end{bmatrix} \tag{11}$$

and

$$n^\lambda = \begin{bmatrix} \frac{1}{N} & \frac{-N^k}{N} \end{bmatrix} \tag{12}$$

where $n^\lambda$ is the unit normal to the hypersurface $\sigma$. When integrating $\mathcal{H}$ over a given hypersurface $\sigma$ the term $(\partial_j \phi \partial^j \phi + m^2 \phi^2)$ may be replaced by a term $\left(K^{\frac{1}{2}}\phi\right)^2$ where[5]

$$K\phi = -\frac{1}{\sqrt{-g}}\partial_k(\sqrt{-g}\partial^k \phi) + m^2 \phi. \tag{13}$$

We assume that $K^{\frac{1}{2}}$ is self-adjoint. Using the usual wave equation

$$\Box \phi + m^2 \phi = 0 \tag{14}$$

one gets

$$K\phi = -\frac{1}{\sqrt{-g}}\partial_0(\sqrt{-g}\partial^0 \phi). \tag{15}$$

In general one does not have separation of space and time without restrictions on the metric. A static or stationary metric does enable such a separation and an analysis in terms of normal modes [7].

## 4. Quantization of the real scalar field

In the representation with $\phi(x)$ a multiplicative operator on a hypersurface $\sigma$ one introduces the momentum operator $\pi$ which satisfies the covariant commutation relation

$$[\phi(x), \pi(y)] = \mathrm{i}\frac{\delta(x-y)}{\sqrt{-g}} \tag{16}$$

(see [7] for details). One has, therefore,

$$\sqrt{-g}\pi(x) = -\mathrm{i}\frac{\delta}{\delta \phi(x)} \tag{17}$$

[3] One can note that in terms of the canonical energy–momentum tensor $T_\tau^\lambda$ one has $(-\frac{\delta I}{\delta \Omega}) = n_\lambda T_\tau^\lambda n^\tau$ which agrees with other authors [5, 6] as to the energy density.

[4] The Latin indices refer to the coordinates on the space-like hypersurface $\sigma$.

[5] See the appendix.

or

$$\pi(x) = -\mathrm{i}\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi(x)}. \tag{18}$$

One should especially note that

$$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi(y)}\int \sqrt{-g}\,\mathrm{d}^3x\, F(x)\phi(x) = F(y). \tag{19}$$

For convenience we set

$$K^{\frac{1}{2}}\phi = \chi \tag{20}$$

and

$$-\mathrm{i}\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi(x)} = -\mathrm{i}\frac{\delta}{\delta\phi_g} \tag{21}$$

and introduce the operators

$$A^{+}(x) = \frac{1}{\sqrt{2}}\left(\chi(x) - \frac{\delta}{\delta\phi_g(x)}\right) \tag{22}$$

$$A^{-}(x) = \frac{1}{\sqrt{2}}\left(\chi(x) + \frac{\delta}{\delta\phi_g(x)}\right). \tag{23}$$

The Hamiltonian is then

$$H = \int \sqrt{-g}\,\mathrm{d}^3x\, A^{+}(x)A^{-}(x). \tag{24}$$

On flat space-like hypersurfaces this reduces to the conventional Hamiltonian [8, 9] and also the Hamiltonian used by Bohm [10]. The commutation relation for $\chi(x)$ and $\mathrm{i}\pi(y)$ is

$$\left[\chi(x), \frac{\delta}{\delta\phi_g(y)}\right]\Psi = -\frac{\delta\chi(x)}{\delta\phi_g(y)}\Psi. \tag{25}$$

Since

$$\chi(x) = \left(K^{\frac{1}{2}}\phi\right)(x) = \int \sqrt{-g}\,\mathrm{d}^3y\, G(x,y)\phi(y) \tag{26}$$

then

$$\frac{\delta\chi(x)}{\delta\phi_g(y)} = G(x,y). \tag{27}$$

One can then write in general

$$\int \sqrt{-g}\,\mathrm{d}^3y\,\frac{\delta\chi(x)}{\delta\phi_g(y)}h(y) = \left(K^{\frac{1}{2}}h\right)(x) \tag{28}$$

for functions $h(y)$ in the domain of $K^{\frac{1}{2}}$. A vacuum state is defined by

$$A^{-}(x)\Psi_0[\phi] = 0 \tag{29}$$

$$\Psi_0[\phi] = \exp\left[-\frac{1}{2}\int \sqrt{-g}\,\mathrm{d}^3x\,\phi(x)\chi(x)\right]. \tag{30}$$

One can now form a set of functionals (not normalized)

$$\Psi_n[\phi; h_1\cdots h_n] = \int\left[\prod_m(\sqrt{-g}\,\mathrm{d}^3x_m)h_m(x_m)\right]A^{+}(x_n)A^{+}(x_{n-1})\cdots A^{+}(x_1)\Psi_0[\phi]. \tag{31}$$

The set of functions $h_1 \cdots h_n$ label the functional and have the same domain as $\phi(x)$. To illustrate this process we generate $\Psi_1[\phi; h_1]$ and $\Psi_2[\phi; h_1, h_2]$. For $\Psi_1[\phi; h_1]$ we have

$$A^+(z_1)\exp\left(-\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^3x\,\phi\chi\right) = \frac{1}{\sqrt{2}}\left(\chi(z_1) - \frac{\delta}{\delta\phi_g(z_1)}\right)\exp\left(-\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^3x\,\phi\chi\right)$$
$$= \sqrt{2}\chi(z_1)\exp\left(-\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^3x\,\phi\chi\right). \qquad (32)$$

Hence,

$$\Psi_1[\phi; h_1] = \sqrt{2}\int\sqrt{-g}\,\mathrm{d}^3z\,h_1(z)\chi(z)\exp\left(-\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^3x\,\phi\chi\right). \qquad (33)$$

For $\Psi_2[\phi; h_1, h_2]$ we have

$$A^+(z_2)A^+(z_1)\Psi_0[\phi] = \frac{1}{\sqrt{2}}\left(\chi(z_2) - \frac{\delta}{\delta\phi_g(z_2)}\right)\chi(z_1)\exp\left(-\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^3x\,\phi\chi\right)$$
$$= \left(\chi(z_2)\chi(z_1) - \frac{\delta\chi(z_1)}{\delta\phi_g(z_2)} + \chi(z_2)\chi(z_1)\right)\exp\left(-\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^3x\,\phi\chi\right). \qquad (34)$$

Hence,

$$\Psi_2[\phi; h_1, h_2] = \int(\sqrt{-g}\,\mathrm{d}^3\,z_1)(\sqrt{-g}\,\mathrm{d}^3\,z_2)A^+(z_2)A^+(z_1)\Psi_0[\phi]$$
$$= 2\left(\int\sqrt{-g}\,\mathrm{d}^3x\,h_2\chi\right)\left(\int\sqrt{-g}\,\mathrm{d}^3x\,h_1\chi\right)\Psi_0[\phi]$$
$$-\left(\int\sqrt{-g}\,\mathrm{d}^3x\,h_1K^{\frac{1}{2}}h_2\right)\Psi_0[\phi]. \qquad (35)$$

As a special case one could consider a hypersurface with a set of orthogonal functions satisfying

$$K^{\frac{1}{2}}\phi_n = \omega_n\phi_n \qquad (36)$$

and then choose the set of $\phi_n$ to specify the functionals. Putting $\phi = \Sigma q_n\phi_n$ one gets

$$\Psi_1[\phi; \phi_n] = \sqrt{2}q_n\omega_n\,\mathrm{e}^{-\frac{1}{2}\Sigma\omega_r q_r^2} \qquad (37)$$

$$\Psi_2[\phi; \phi_n, \phi_m] = [2q_nq_m\omega_n\omega_m - \omega_m\delta_{nm}]\,\mathrm{e}^{-\frac{1}{2}\Sigma\omega_r q_r^2}. \qquad (38)$$

These are proportional to the usual Hermite functions of the mode coordinates $q_n$.

## 5. Action of the Hamiltonian density on a functional

Taking the Hamiltonian density from (24) and applying it to the functional of (31).

$$A^+(x)A^-(x)[A^+(x_n)A^+(x_{n-1})\cdots A^+(x_1)\Psi_0[\phi]] = A^+(x)\left[A^+(x_n)A^-(x) + \frac{\delta\chi(x)}{\delta\phi_g(x_n)}\right]$$
$$\times A^+(x_{n-1})\cdots A^+(x_1)\Psi_0[\phi].$$

Repeating the process until $A^-(x)$ operates on $\Psi_0[\phi]$ one gets

$$\frac{\delta\chi(x)}{\delta\phi_g(x_n)}A^+(x)A^+(x_{n-1})\cdots A^+(x_1)\Psi_0[\phi]$$
$$+\frac{\delta\chi(x)}{\delta\phi_g(x_{n-1})}A^+(x)A^+(x_n)A^+(x_{n-2})\cdots A^+(x_1)\Psi_0[\phi] + \cdots.$$

Multiplying this last expression by $h_n(x_n)\cdots h_1(x_1)$ and integrating over $x, x_n, x_{n-1}, \ldots, x_1$ one gets

$$\Sigma_m \left( \int \sqrt{-g}\, \mathrm{d}^3x \left(K^{\frac{1}{2}} h_m\right)(x) A^+(x) \right) \Psi_{n-1}[\phi; h_n h_{n-1} \cdots h_{m+1}, h_{m-1} \cdots h_1]. \tag{39}$$

We illustrate the above with two examples of $\mathcal{H}$ applied to $\Psi_1[\phi; h_1]$ and $\Psi_2[\phi; h_1, h_2]$. Firstly, for $\Psi_1[\phi; h_1]$ we have

$$\begin{aligned} A^+(x)A^-(x)[A^+(x_1)\Psi_0[\phi]] &= A^+(x)\left[A^+(x_1)A^-(x) + \frac{\delta\chi(x)}{\delta\phi_g(x_1)}\right]\Psi_0[\phi] \\ &= \frac{\delta\chi(x)}{\delta\phi_g(x_1)} A^+(x)\Psi_0[\phi] \\ &= \frac{\delta\chi(x)}{\delta\phi_g(x_1)} \sqrt{2}\chi(x)\Psi_0[\phi]. \end{aligned}$$

Multiplying by $h_1(x_1)$ and integrating over $x$ and $x_1$ one gets

$$\sqrt{2}\left[\int \sqrt{-g}\, \mathrm{d}^3x\, \chi K^{\frac{1}{2}} h_1\right]\Psi_0[\phi]. \tag{40}$$

Secondly, for $\Psi_2[\phi; h_1, h_2]$ we have

$$\begin{aligned} A^+(x)A^-(x)[A^+(x_2)A^+(x_1)\Psi_0[\phi]] &= \left[\frac{\delta\chi(x)}{\delta\phi_g(x_1)} A^+(x)A^+(x_1) + \frac{\delta\chi(x)}{\delta\phi_g(x_1)} A^+(x)A^+(x_2)\right]\Psi_0[\phi] \\ = \frac{\delta\chi(x)}{\delta\phi_g(x_2)}&\left[2\chi(x)\chi(x_1) - \frac{\delta\chi(x_1)}{\delta\phi_g(x)}\right]\Psi_0[\phi] \\ + \frac{\delta\chi(x)}{\delta\phi_g(x_1)}&\left[2\chi(x)\chi(x_2) - \frac{\delta\chi(x_2)}{\delta\phi_g(x)}\right]\Psi_0[\phi] \end{aligned} \tag{41}$$

multiplying by $h_1(x_1)h_2(x_2)$ and integrating over $x$, $x_1$ and $x_2$ one gets

$$\begin{aligned} 2&\left[\int \sqrt{-g}\, \mathrm{d}^3x\, \chi K^{\frac{1}{2}} h_2\right]\left[\int \sqrt{-g}\, \mathrm{d}^3x\, \chi h_1\right]\Psi_0[\phi] \\ &+2\left[\int \sqrt{-g}\, \mathrm{d}^3x\, \chi K^{\frac{1}{2}} h_1\right]\left[\int \sqrt{-g}\, \mathrm{d}^3x\, \chi h_2\right]\Psi_0[\phi] \\ &-\left[\int \sqrt{-g}\, \mathrm{d}^3x (h_2)(Kh_1)\right]\Psi_0[\phi] - \left[\int \sqrt{-g}\, \mathrm{d}^3x (Kh_2)(h_1)\right]\Psi_0[\phi]. \end{aligned} \tag{42}$$

In general one will have to integrate (2) numerically since the lapse function $(N)$ for the foliation may involve the scalar time parameter $t$. Once a solution $\Psi[\sigma]$ is obtained, the evolution of the Bohm field $\phi_{\mathrm{Bohm}}$ on the given foliation $\sigma$ is determined by the usual Bohmian guidance condition [10]

$$\frac{1}{N}\frac{\mathrm{d}\phi_{\mathrm{Bohm}}}{\mathrm{d}t} = \mathrm{Im}\left[\frac{1}{\Psi^{(t)}[\phi]}\frac{\delta\Psi^{(t)}[\phi]}{\delta\phi_g}\right] \tag{43}$$

where successive leaves are labelled by the parameter $t$. In a de Broglie–Bohm type of theory the foliation should not be subject to an arbitrary choice but rather be determined by the physical properties of the system itself. (See section (7) for further details.)

## 6. Integration of the 'Schrödinger' equation for static and stationary metric

In the case of a static or stationary metric one easily obtains the integrated wave functional in the form

$$\Psi^{(t=0)}[\phi] = \sum a_n \Psi_n\big[\phi; h_1^{(n)} \ldots, h_n^{(n)}\big] \tag{44}$$

which evolves to

$$\Psi^{(t)}[\phi] = \sum a_n \Psi_n \left[\phi; \mathrm{e}^{-\mathrm{i}K^{\frac{1}{2}}t} h_1^{(n)} \ldots, \mathrm{e}^{-\mathrm{i}K^{\frac{1}{2}}t} h_n^{(n)}\right] \tag{45}$$

$\phi$ is the time independent scalar field of the Schrödinger picture. The result follows from the comparison of $\mathrm{i}\frac{\mathrm{d}\Psi[\phi]}{\mathrm{d}t}$ and equation (39) which gives the integral $\int_\sigma \sqrt{-g}\,\mathrm{d}^3x\,\mathcal{H}(x)\Psi$.

To illustrate the procedure consider a family of space-like hypersurfaces with a set of orthogonal functions $\phi_n(x)$ such that

$$K^{\frac{1}{2}}\phi_n(x) = \omega_n\phi_n(x). \tag{46}$$

Let

$$h = \sum b_n\phi_n(x) \tag{47}$$

and

$$\phi = \sum c_n\phi_n(x). \tag{48}$$

Suppose that at $t = 0$

$$\Psi^{(t=0)}[\phi] = a_0\Psi_0[\phi] + a_1\Psi_1[\phi; h]. \tag{49}$$

Then

$$\Psi^{(t)}[\phi] = a_0\Psi_0[\phi] + \sqrt{2}a_1\left[\int \sqrt{-g}\,\mathrm{d}^3x\,\chi\,\mathrm{e}^{-K^{\frac{1}{2}}t}h\right]\Psi_0[\phi] \tag{50}$$

where

$$\chi = K^{\frac{1}{2}}\phi = \sum c_n\omega_n\phi_n(x). \tag{51}$$

Therefore,

$$\Psi^{(t)}[\phi] = a_0\Psi_0[\phi] + \sqrt{2}a_1\left[\sum b_n c_n\omega_n\,\mathrm{e}^{-\mathrm{i}\omega_n t}\right]\Psi_0[\phi] \tag{52}$$

$\phi_{\mathrm{Bohm}}$ will be time dependent so we write

$$\phi_{\mathrm{Bohm}} = \sum q_n(t)\phi_n. \tag{53}$$

Noting that from the definition of the vacuum, (23), (29)

$$\frac{\delta\Psi_0[\phi]}{\delta\phi_g(x)} = -\chi(x)\Psi_0[\phi] \tag{54}$$

we have

$$\left[\frac{1}{\Psi^{(t)}[\phi]}\frac{\delta\Psi^{(t)}[\phi]}{\delta\phi_g}\right] = -\chi(x) + \frac{\sqrt{2}a_1\big[\sum b_n\omega_n\,\mathrm{e}^{-\mathrm{i}\omega_n t}\phi_n(x)\big]}{a_0 + \sqrt{2}a_1\big[\sum b_n c_n\omega_n\,\mathrm{e}^{-\mathrm{i}\omega_n t}\big]}. \tag{55}$$

Since in this example $\chi$ is a real field, the guidance condition, equation (43), yields

$$\frac{1}{N}\frac{\mathrm{d}\phi_{\mathrm{Bohm}}}{\mathrm{d}t} = \mathrm{Im}\left[\frac{1}{\Psi^{(t)}[\phi]}\frac{\delta\Psi^{(t)}[\phi]}{\delta\phi_g}\right] = \mathrm{Im}\left[\frac{\sqrt{2}a_1\big[\sum b_n\omega_n\,\mathrm{e}^{-\mathrm{i}\omega_n t}\phi_n(x)\big]}{a_0 + \sqrt{2}a_1\big[\sum b_n c_n\omega_n\,\mathrm{e}^{-\mathrm{i}\omega_n t}\big]}\right]. \tag{56}$$

The $c_n$ in equation (56) must, of course, be the coefficients giving $\phi_{\mathrm{Bohm}}$ at time $t$.

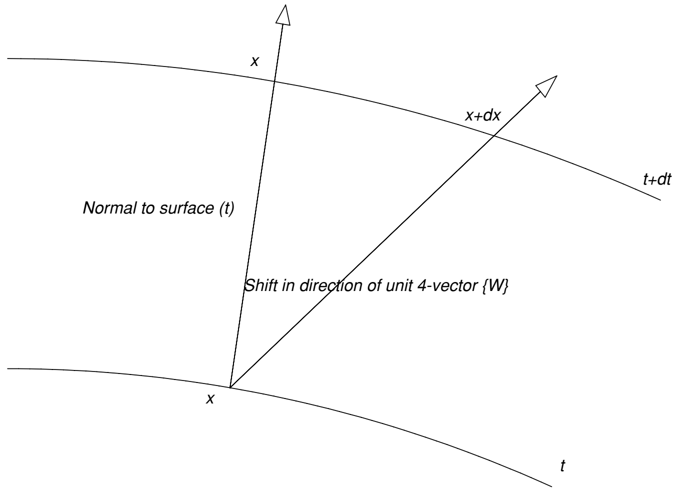


**Figure 1.** The construction of the foliation.

## 7. Beables of the field

In constructing a theory of 'beables'[6] one is free to choose the initial space-like hypersurface and field $\phi_{\rm Bohm}$ thereon. Each choice corresponds to a different state of affairs. However, in integrating equation (43) one needs to give an invariant rule constructing the next leaf of the foliation in order to get a definite $\phi_{\rm Bohm}$. One restriction is to choose to displace each point on the initial surface by equal proper times; the direction of the displacement will also need to be specified by a unit time-like 4-vector $[W_\mu]$. The construction of the foliation is shown in figure 1. The shift vectors are chosen to be zero as other choices only correspond to different coordinate systems on the surface

$$ds^2 = N^2\,dt^2 - dx^i g_{ij}\,dx^j. \tag{57}$$

The three velocity of the shift given by $[W^\mu]$ is

$$v^i = \frac{W^i}{W^0} \tag{58}$$

the proper time along the shift vector is $d\epsilon$, constant for all points on the surface and the proper time along the normal is $N\,dt$. Therefore,

$$N\,dt = d\epsilon\cosh\theta \tag{59}$$

with

$$\cosh\theta = \frac{1}{\sqrt{1-v^2}}. \tag{60}$$

It is convenient to choose $N = \cosh\theta$ so that $dt = d\epsilon$. In one spatial dimension one also has

$$g_{11} = \sinh\theta = \frac{v}{\sqrt{1-v^2}} \tag{61}$$

or, in general

$$dx^i g_{ij}\,dx^j = \left(\frac{v}{\sqrt{1-v^2}}\,d\epsilon\right)^2 \tag{62}$$

[6] The concept of beables was introduced by Bell [11].

for displacements along the vector $[W^\mu]$. If

$$\frac{1}{N}\frac{\mathrm{d}\phi_{\mathrm{Bohm}}}{\mathrm{d}t} = \frac{\delta S}{\delta\phi_g} \tag{63}$$

on one leaf of the foliation at $t$ then on the leaf $t + \mathrm{d}t$ *at the same coordinate point* $[x^i]$

$$\Delta\phi = \frac{\mathrm{d}\phi}{\mathrm{d}t}\,\mathrm{d}\epsilon = N\frac{\delta S}{\delta\phi_g}\,\mathrm{d}\epsilon. \tag{64}$$

One has interpreted $\frac{1}{N}\frac{\mathrm{d}\phi}{\mathrm{d}t}$ as the momentum and, writing $\Psi[\phi] = R[\phi]\,\mathrm{e}^{\mathrm{i}S[\phi]}$, replaced it by $\frac{\delta S}{\delta\phi_g}$ as in Bohm's original theory.

One possible choice of $W^\mu$ is to follow the rule we have previously adopted in the many-particle case and use the time-like eigenvector of the energy–momentum tensor $T_{\mu\nu}$ [12–14]. The $T_{\mu\nu}$ will be that of a classical scalar field $\phi$ but with $\phi$ replaced by $\phi_{\mathrm{Bohm}}$.

## Appendix

$$\int\sqrt{-g}\,\mathrm{d}^3x\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}(\sqrt{-g}\phi\partial^k\phi)\right] = \int\mathrm{d}^3x\,\frac{\partial}{\partial x^k}(\sqrt{-g}\phi\partial^k\phi). \tag{65}$$

This expression can be converted into an integral over a bounding surface which will vanish for suitable behaviour of $\sqrt{-g}\phi\partial^k\phi$. Hence,

$$\int\sqrt{-g}\,\mathrm{d}^3x\left[\partial_k\phi\partial^k\phi + \phi\left(\frac{1}{\sqrt{-g}}\partial_k(\sqrt{-g}\partial^k\phi)\right)\right] = 0 \tag{66}$$

and

$$\int\sqrt{-g}\,\mathrm{d}^3x[\partial_k\phi\partial^k\phi] = -\int\sqrt{-g}\,\mathrm{d}^3x\left[\phi\left(\frac{1}{\sqrt{-g}}\partial_k(\sqrt{-g}\partial^k\phi)\right)\right]. \tag{67}$$

## References


[1] Duerr D *et al* 1999 *Phys. Rev.* A **60** 2729
[2] Schwinger J 1948 *Phys. Rev.* **74** 1439
[3] Barut A O 1980 *Electrodynamics and Classical Theory of Fields and Particles* (New York: Dover) p 103
[4] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (New York: Freeman)
[5] Schutz B F 1971 *Phys. Rev.* D **4** 3559
[6] deWet J S 1950 *Proc. Royal Soc. London* A **201** 284
[7] Fulling S A 1989 *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge: Cambridge University Press)
[8] Schweber S S 1961 *An Introduction to Relativistic Quantum Field Theory* (New York: Harper and Row)
[9] Rider L H 1985 *Quantum Field Theory* (Cambridge: Cambridge University Press)
[10] Bohm D, Hiley B J and Kaloyerou P 1987 *Phys. Rep.* **144** 349
[11] Bell J S 1987 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press)
[12] Horton G, Dewdney C and Nesteruk A 2000 *J. Phys. A: Math. Gen.* **33** 7337
[13] Horton G and Dewdney C 2001 *J. Phys. A: Math. Gen.* **34** 9871
[14] Dewdney C and Horton G 2002 *J. Phys. A: Math. Gen.* **35** 10017